Theory of second- and higher-order stochastic processes

Alon Drory

Center for Polymer Studies and Department of Physics, Boston University, Boston, Massachusetts 02215

This paper presents a general approach to linear stochastic processes driven by various random noises. Mathematically, such processes are described by linear stochastic differential equations of arbitrary order [the simplest nontrivial example is $\ddot{x} = R(t)$, where $R(t)$ is not a Gaussian white noise]. The stochastic process is discretized into n time steps, all possible realizations are summed up, and the continuum limit is taken. This procedure often yields closed form formulas for the joint probability distributions. Completely worked out examples include all Gaussian random forces and a large class of Markovian (non-Gaussian) forces. This approach is also useful for deriving Fokker-Planck equations for the probability distribution functions. This is worked out for Gaussian noises and for the Markovian dichotomous noise.

PACS number(s): 05.40.+j, 02.50.−r

I. INTRODUCTION

Stochastic differential equations were first studied in the context of Brownian motion by Langevin. For a particle of unit mass, he derived the equation for the displacement

$$\ddot{x} + \gamma \dot{x} = R(t) . \tag{1.1}$$

In this equation, the drag force $-\gamma \dot{x}$ is a deterministic term which represents the "environment" in which the particle moves. In the case of Brownian motion, this is the fluid surrounding the particle. The force driving the particle through this environment is the random term $R(t)$ which is assumed to be a Gaussian white noise [1].

In recent years, the field of stochastic equations has expanded considerably along two main lines, each focusing on one of the two basic aspects of Langevin equations: The first concentrates on more complex environments, usually introduced through potentials $U(x)$; the second concentrates on random driving forces more complex than the Gaussian white noise.

The first approach deals with equations like, e.g.,

$$\ddot{x} + \frac{dU(x)}{dx} = R(t) , \tag{1.2}$$

where $U(x)$ is an external potential [2]. Since Eq. (1.2) is typically nonlinear and very hard to solve, the random noise is usually kept Gaussian and white. [As an aside, note that when one tries to combine nontrivial potential terms with colored noises [3–5] the difficulties are so great that the order of the equation has to be reduced to the form [4] $\dot{x} + \Omega(x) = R(t)$.]

The present work, however, follows the second line of generalizations of Eq. (1.1), which leaves the "environment" relatively simple but considers more complex random forces without reducing the order of the equation. It has been known for some time, for example, that a more realistic description of Brownian motion must include a finite correlation time for the random force [6], a

feature absent in the Gaussian white noise. If the random force arises from processes internal to the system (as in Brownian motion), the system may reach a state of detailed balance at equilibrium. In this case, if the force is not Gaussian white, the friction term requires a modification which usually involves a retardation effect [7]. Equation (1.1) is then generalized to be

$$\ddot{x} + \int_{-\infty}^{t} \gamma(t - t')\dot{x}(t')dt' = R(t) , \tag{1.3}$$

where $\gamma(t - t')$ is determined by the correlation function $\langle R(t)R(t') \rangle$ (hereafter, angular brackets denote average values) [7]. Unlike the generalization Eq. (1.2), Eq. (1.3) is linear in the stochastic variable $x(t)$. Nonetheless, it is usually still difficult to solve. Often, it is simplified by assuming the friction to be negligible. (This cannot be done if detailed balance is required. However, in many systems, this requirement is not necessary because the force is an external influence.) One then ends up with the simple equation

$$\ddot{x} = R(t) . \tag{1.4}$$

This approximation has been used recently to describe some reaction-diffusion processes in a force-dominated regime [8].

Even the simple equation (1.4) is still hard to solve when the random force is more complex than Gaussian white noise. In this context, solving the equation means finding the joint probability distribution function (PDF) for the position and the velocity, $p(x,v,t)$, the marginal distribution for the velocity, $p(v,t) \equiv \int p(x,v,t)dx$, and the marginal distribution for the position, $p(x,t) \equiv \int p(x,v,t)dv$. Equation (1.4) does not relate to these functions directly, however, and the usual approach is therefore to obtain a Fokker-Planck (FP) equation for the PDF's. Going from a stochastic equation like (1.4) to a FP equation can be quite difficult. After this is done, one still has to solve the resulting FP equation, which is far from trivial [9–11].

In the present paper, I outline a general approach to

linear stochastic equations [including Eqs. (1.3) and (1.4)], that often yields a closed form expression for the various PDF's, and may also simplify the derivation of Fokker-Planck equations. This approach is based on a *discretization* of the stochastic process represented by the Langevin equation by breaking up the continuous time variable into a finite number of steps, n. The discrete process thus generated is completely described by an n-order distribution function W_n. In Sec. II, I derive the fundamental formula of this paper, which yields $p(x,v,t)$ as the limit $n \to \infty$ of an expression involving W_n. I do not attempt a rigorous mathematical proof that the limit exists in general. This would be an extremely difficult problem since the expression obtained in Sec. II is basically a path integral. However, the path-integral formalism is entirely bypassed in this work in favor of a direct calculation approach. Indeed, the remaining sections of the paper are devoted to specific examples where the calculations can be carried through without using any path-integral method.

Thus, in Sec. III, the fundamental formula is applied to a class of Markov noises termed *monovariant*. This includes the Ornstein-Uhlenbeck noise, as well as, e.g., the Wiener and Cauchy processes. In Sec. IV, the same formula yields a closed expression for the various PDF's when the noise is any Gaussian process (not necessarily Markovian). Finally, I use the general formula to derive FP-like equations: Sec. V treats Gaussian forces, and Sec. VI treats the Markovian dichotomous noise, with which many investigations were concerned in recent years. Section VII summarizes the approach and its main results.

II. GENERAL FORMALISM

Equations (1.3) and (1.4) are particular cases of the general equation

$$L(x,t) = R(t) , \qquad (2.1)$$

where $L(x,t)$ is an operator acting on the stochastic variable x. I assume that $L(x,t)$ is such that the general solution of the equation can be written formally as

$$x = u(t) + \int_0^t g(t,t')R(t')dt' . \qquad (2.2)$$

$u(t)$ is the solution to the homogeneous equation and $g(t,t')$ is the Green's function of the equation. Often, $g(t,t')$ depends only on the combination $t - t'$, but this is not yet assumed.

Clearly, a similar relation holds for the velocity as well. Thus we can find functions $w(t)$ and $h(t,t')$ such that

$$v(t) = w(t) + \int_0^t h(t,t')R(t')dt' . \qquad (2.3)$$

Equations (2.2) and (2.3), rather than Eq. (2.1), are the starting point of our approach. Thus, in the remainder of this paper, I assume that $u(t)$, $w(t)$, $h(t,t')$, and $g(t,t')$ are given functions (though they may be hard to determine in practice). For simplicity, I also assume throughout the paper that $\langle R(t) \rangle = 0$. The modifications required to accommodate a nonzero mean value are straightforward but tedious.

The number of stochastic variables of interest is usually determined by the order of the stochastic equation. Thus, in a second-order process such as described by $\ddot{x} = R(t)$, we have two variables of interest, $x(t)$ and $v(t)$. However, the method described here is general and applicable to any number of stochastic variables and therefore to equations of arbitrary order. However, for definiteness, I will describe it in the context of second-order processes, which are the most common in physics.

The main idea presented here is to transform the continuous process described by Eqs. (2.2) and (2.3) into a discrete process involving finite time steps Δt. Such discrete processes can be described easily through the n-order distribution function of the random noise $R(t)$. It turns out that the Fourier transform of this distribution function relates very simply to the Fourier transform of the PDF of the stochastic variable. This basic relation can be used to calculate the PDF explicitly in some cases, or to easily derive Fokker-Planck-like equations in other cases.

To derive this fundamental relation, note that the discretized version of the random noise $R(t)$ is fully determined by the set of all its n-order distribution functions

$$W_n(y_1,t_1;y_2,t_2; \ldots ;y_n,t_n)dy_1 \cdots dy_n$$

$$\equiv \mathrm{Prob}\{y_i < R(t_i) < y_i + dy_i \text{ for all } i = 1, \ldots, n \} ,$$

$$(2.4)$$

i.e.,

$$W_n(y_1,t_1;y_2,t_2; \ldots ;y_n,t_n)dy_1 \cdots dy_n$$

is the probability that at the times t_i the value of the random noise $R(t_i)$ is in the range $(y_i, y_i + dy_i)$. The functions W_n define the full process $R(t)$ through discrete and finite sets of times $\{t_i\}_{i=1}^n$. This suggests describing the random variables $x(t)$ and $v(t)$ through similar finite sets of times. To this end, define a time step $\Delta t = t/n$, where n is an integer that will eventually go to infinity.

Consider now a *particular* realization of the process, i.e., a specific set of force values $\{y_i = R(i\Delta t)\}_{i=1}^n$. For this realization, define sets $\{x_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$ through a discretization of Eqs. (2.2) and (2.3), i.e.,

$$x_n = u_n + \sum_{j=1}^n g(n,j)y_j\Delta t , \qquad (2.5a)$$

$$v_n = w_n + \sum_{j=1}^n h(n,j)y_j\Delta t , \qquad (2.5b)$$

where

$$u_n = u(n\Delta t), \quad w_n = w(n\Delta t) , \qquad (2.5c)$$

$$g(n,j) = g(n\Delta t, j\Delta t), \quad h(n,j) = h(n\Delta t, j\Delta t) . \qquad (2.5d)$$

As $n \to \infty$ and $\Delta t \to 0$ with $n\Delta t$ remaining constant, the sets $\{x_i\}$ and $\{v_i\}$ converge to the continuous processes $x(t)$ and $v(t)$.

Let us now define a discrete analog of the joint PDF $p(x,v,t)$. This discrete PDF, denoted $p(x,v,n)$, is defined as

$$p(x,v,n)dx\,dv = \text{Prob}\{x < x_n < x + dx$$
$$\text{and } v < v_n < v + dv\}\ , \qquad (2.6)$$

where x_n and v_n are now *defined* by Eqs. (2.4) and (2.6), with respect to the values y_i of the random noise $R(t)$. Our aim is to calculate $p(x,v,n)$ explicitly and hope that, as $n \to \infty, \Delta t \to 0, p(x,v,n)$ will tend to $p(x,v,t)$.

The probability $p(x,v,n)$ is the total probability of all possible realizations $\{y_i\}$ such that $x < x_n < x + dx$ and $v < v_n < v + dv$, where x_n and v_n are given by Eqs. (2.4)–(2.6) and $n = t/\Delta t$. For simplicity, I assume that the set $\{y_i\}_{i=1}^n$ may take any value in the range $(-\infty, \infty)$ (if the range is discrete, one can use appropriate δ functions to restrict the values of y_i). Therefore we have

$$p(x,v,n) = \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n\, W_n(y_1, \Delta t; y_2, 2\Delta t; \ldots; y_n, n\Delta t)\delta(x - x_n)\delta(v - v_n)\ . \qquad (2.7)$$

The two δ functions in the integral, $\delta(x - x_n)$ and $\delta(v - v_n)$, ensure that only processes $\{y_i\}_{i=1}^n$ which yield the appropriate final positions and velocities are counted. Note that x_n and y_n are functions of $\{y_i\}$ through the relations (2.5). For simplicity, I shall write from now on $W_n(y_1, \ldots, y_n)$ instead of $W_n(y_1, \Delta t; y_2, 2\Delta t; \ldots; y_n, n\Delta t)$.

Equation (2.7) can be simplified by introducing the Fourier transforms (FT),

$$\hat{p}(k,\theta,n) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv\, e^{i(xk + v\theta)}p(x,v,n)\ , \qquad (2.8a)$$

$$\hat{W}(\varphi_1, \ldots, \varphi_n)$$
$$= \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n \exp[i(y_1\varphi_1 + \cdots + y_n\varphi_n)]$$
$$\times W_n(y_1, \ldots, y_n)\ . \qquad (2.8b)$$

Substituting Eq. (2.7) into Eq. (2.8) and interchanging the order of integration, we obtain

$$\hat{p}(k,\theta,n) = \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv\, e^{(xk + v\theta)}W_n(y_1, \ldots, y_n)\delta(x - x_n)\delta(v - v_n)\ . \qquad (2.9)$$

Performing the integration over x and v and replacing x_n and v_n with the corresponding expressions from Eqs. (2.5) which relate them to the values y_i, we have

$$\hat{p}(k,\theta,n) = \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n\, e^{(u_n k + w_n \theta)}W_n(y_1, \ldots, y_n)\exp\left\{i\sum_{j=1}^{n}[kg(n,i) + \theta h(n,i)]y_i\Delta t\right\}\ . \qquad (2.10)$$

Defining

$$\varphi_i = [g(n,i)k + h(n,i)\theta]\Delta t\ ,$$

and comparing with Eq. (2.8b), we see that

$$\hat{p}(k,\theta,n) = e^{i(u_n k + w_n \theta)}\hat{W}(\varphi_1, \ldots, \varphi_n)\ , \qquad (2.11a)$$

where

$$\varphi_i = [g(n,i)k + h(n,i)\theta]\Delta t\ . \qquad (2.11b)$$

Ultimately, however, we are interested in the FT of $p(x,v,t)$,

$$\hat{p}(k,\theta,t) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv\, e^{i(xk + v\theta)}p(x,v,t)\ . \qquad (2.12)$$

To obtain this, we take the limit $n \to \infty, \Delta t \to 0$ of $\hat{p}(k,\theta,n)$, i.e.,

$$\hat{p}(k,\theta,t) = \lim_{\substack{n \to \infty \\ \Delta t \to 0 \\ n\Delta t = t}} e^{i(u_n k + w_n \theta)}\hat{W}(\varphi_1, \ldots, \varphi_n)\ , \qquad (2.13a)$$

with

$$\varphi_i = [g(n,i)k + h(n,i)\theta]\Delta t\ . \qquad (2.13b)$$

Equation (2.13) is the central result of this section and the basis for the rest of this paper, which deals with specific applications.

Because of its importance, one should note that, while reasonable from the physical point of view, Eq. (2.13) nonetheless raises some subtle mathematical questions. We cannot prove that the limit $n \to \infty, \Delta t \to 0$ of $\hat{p}(k,\theta,n)$ exists in general, nor that, given its existence, it is indeed equal to $\hat{p}(k,\theta,t)$. The source of this difficulty is Eq. (2.7), which is clearly a discrete path integral, and therefore subject to various mathematical reservations when the limit $n \to \infty$ is taken formally.

However, in this paper the usual formalism associated with path integrals is bypassed in favor of the direct definition Eq. (2.13). In other words, we do not take the limit $n \to \infty, \Delta t \to 0$ formally and write the resulting expression as a path integral. Rather, all calculations are carried out for the finite n case, and the limit then taken directly. It is remarkable that in many cases this limit turns out to be well defined mathematically as well as calculable, with the only assumptions being the smoothness properties usually postulated in physical problems. The next two sections are devoted to such calculations and cover a wide range of cases.

III. MONOVARIANT MARKOV NOISE PROCESSES

As a first example where the limit in Eq. (2.13) can be calculated explicitly, consider a class of Markov noise processes defined below.

A Markov process is fully determined by the initial dis-

tribution $P_0(y_1)$, and the transition probability $T(y_{i+1}, t_i + \Delta t | y_i, t_i)$ from the value y_i of the random function $R(t_i)$ at time t_i to the value $y_{i+1} = R(t_i + \Delta t)$ at a time $t_i + \Delta t$ [12]. It is usually assumed that this function depends only on the time step Δt and not on t_i. I shall assume this for simplicity, but the calculations can be modified for more general cases.

I shall say a Markov process is *monovariant* if the transition probability does not depend on y_{i+1} and y_i separately, but rather on a single linear combination of these variables. Such a combination can always be written as $y_{i+1} - \mu(\Delta t) y_i$, where $\mu(\Delta t)$ is some function of the time step Δt, and I also assume that $\mu(0) \neq 0$. Hence

$$T(y_{i+1}, \Delta t | y_i) = T_{\Delta t}(y_{i+1} - \mu(\Delta t) y_i) . \tag{3.1}$$

For any Markov process, the functions $W_n(y_1, \ldots, y_n)$ are given by [12]:

$$W_n(y_1, \ldots, y_n)$$
$$= P_0(y_1) T_{\Delta t}(y_2 | y_1) T_{\Delta t}(y_3 | y_2) \cdots T_{\Delta t}(y_n | y_{n-1}) , \tag{3.2}$$

where we assume all transitions take place during a time step $\Delta t = t/n$, with n eventually going to infinity.

Many of the noises used in physical problems are monovariant. Some examples are the Wiener noise [with $\mu(\Delta t) = 1$], the Cauchy noise [also with $\mu(\Delta t) = 1$], and the Ornstein-Uhlenbeck (OU) noise, for which $\mu(\Delta t) = \exp(-\Delta t / \tau)$, where τ is the correlation time [12]. The OU noise is widely used to describe forces with finite correlation time (e.g., Ref. [9]). The $\tau \to 0$ limit corresponds to white noise.

Let us now calculate the PDF for a process driven by a general monovariant Markov noise. The multivariable FT of $W_n(y_1, \ldots, y_n)$ is

$$\hat{W}(\varphi_1, \ldots, \varphi_n) = \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n \exp[i(y_1\varphi_1 + \cdots + y_n\varphi_n)] P_0(y_1) T_{\Delta t}(y_2 | y_1) T_{\Delta t}(y_3 | y_2) \cdots T_{\Delta t}(y_n | y_{n-1}) . \tag{3.3}$$

If the Markov noise is monovariant, so that $T_{\Delta t}(y'|y) = T_{\Delta t}(y' - \mu(\Delta t)y)$, we can perform the integration in the following way. Define a single-variable Fourier transform $\hat{T}$ such that

$$\hat{T}_{\Delta t}(\xi) = \int_{-\infty}^{\infty} e^{i\xi\eta} T_{\Delta t}(\eta) d\eta , \tag{3.4}$$

where $\eta = y' - \mu(\Delta t)y$ is the single variable on which $T_{\Delta t}(y'|y)$ depends, except for any direct dependence on Δt. For conciseness, I assume such a dependence implicitly and do not write the Δt subscript anymore. We now change the integration variables $\{y_i\}_{i=1}^{n}$ in Eq. (3.3) to the new set

$$\eta_1 = y_1 ,$$
$$\eta_2 = y_2 - \mu(\Delta t) y_1 ,$$
$$\eta_3 = y_3 - \mu(\Delta t) y_2 , \tag{3.5}$$
$$\vdots$$
$$\eta_n = y_n - \mu(\Delta t) y_{n-1} .$$

The Jacobian of this transformation is clearly unity, and each $T_{\Delta t}(y_i | y_{i-1})$ depends on η_i only. To rewrite $\sum_{i=1}^{n} y_i \varphi_i$ in terms of the new variables, we invert Eq. (3.5), which gives

$$y_1 = \eta_1 ,$$
$$y_2 = \eta_2 + \mu(\Delta t)\eta_1 ,$$
$$y_3 = \eta_3 + \mu(\Delta t)\eta_2 + [\mu(\Delta t)]^2 \eta_1 , \tag{3.6}$$
$$\vdots$$
$$y_i = \sum_{k=1}^{i} [\mu(\Delta t)]^{k-1} \eta_{i-k+1} .$$

Hence,

$$\sum_{i=1}^{n} y_i \varphi_i = \sum_{i=1}^{n} \eta_i \left\{ \sum_{k=i}^{n} \varphi_k [\mu(\Delta t)]^{k-i} \right\} . \tag{3.7}$$

Substituting Eq. (3.7) into the expression for $\hat{W}$, Eq. (3.3), and using the definition of $\hat{T}$, Eq. (3.4), we finally obtain

$$\hat{W}(\varphi_1, \ldots, \varphi_n) = \hat{P}_0(\zeta) \prod_{i=2}^{n} \hat{T}(\xi_i) , \tag{3.8a}$$

with

$$\zeta \equiv \sum_{k=1}^{n} \varphi_k [\mu(\Delta t)]^{k-1} , \tag{3.8b}$$

$$\xi_i \equiv \sum_{k=i}^{n} \varphi_k [\mu(\Delta t)]^{k-i} . \tag{3.8c}$$

In Eq. (3.8a),

$$\hat{P}_0(\zeta) = \int_{-\infty}^{\infty} e^{i\zeta y} P_0(y) dy . \tag{3.9}$$

In accordance with the general formula (2.11), $\hat{p}(k, \theta, n)$ is obtained by replacing φ_i in Eqs. (3.8) with $[g(n,i)k + h(n,i)\theta]\Delta t$. This yields

$$\hat{p}(k, \theta, n) = e^{i(u_n k + w_n \theta)} \hat{P}_0(\zeta) \prod_{i=2}^{n} \hat{T}(\xi_i) , \tag{3.10a}$$

where

$$\zeta \equiv k \left[\sum_{k=1}^{n} g(n,k)[\mu(\Delta t)]^{k-1} \Delta t \right]$$
$$+ \theta \left[\sum_{k=1}^{n} h(n,k)[\mu(\Delta t)]^{k-1} \Delta t \right] , \tag{3.10b}$$

$$\xi_i \equiv k \left[\sum_{k=i}^{n} g(n,k)[\mu(\Delta t)]^{k-i}\Delta t \right]$$
$$+ \theta \left[\sum_{k=i}^{n} h(n,k)[\mu(\Delta t)]^{k-i}\Delta t \right] . \qquad (3.10c)$$

Last, we must take the limit $n \to \infty, \Delta t \to 0$. This is a somewhat tedious step, and its details can be found in Appendix A. The final result is the following very general formula:

$$\hat{p}(k,\theta,t) = e^{i[ku(t)+\theta w(t)]}\hat{P}_0(\zeta(t))$$
$$\times \exp \left\{ \int_0^t dt' B(\xi(t,t')) \right\} , \qquad (3.11a)$$

where

$$B(\xi) = \frac{d}{d\Delta t}[\ln \hat{T}_{\Delta t}(\xi)]_{\Delta t=0} , \qquad (3.11b)$$

$$\zeta(t) \equiv k \int_0^t g(t,s)\Psi(s)ds + \theta \int_0^t h(t,s)\Psi(s)ds , \qquad (3.11c)$$

$$\xi(t,t') \equiv k \int_{t'}^t g(t,s)\Psi(s-t')ds$$
$$+ \theta \int_{t'}^t h(t,s)\Psi(s-t')ds , \qquad (3.11d)$$

with

$$\Psi(z) = \kappa(0)\exp[az] , \qquad (3.11e)$$

$$a \equiv \frac{d[\ln\mu(t)]}{dt}\bigg|_{t=0} = \frac{1}{\mu(t)}\frac{d\mu(t)}{dt}\bigg|_{t=0} . \qquad (3.11f)$$

Equations (3.11) are general and can be applied for a great variety of noise processes and of dynamics (i.e., forms of the Langevin equation). The price of this flexibility is the complex form of the equations. Their meaning and use can be clarified, however, by using as an example a very simple equation which was investigated recently [8,9],

$$\ddot{x} = \dot{v} = R(t) . \qquad (3.12)$$

The solution is $v = v_0 + \int_0^t R(t')dt'$, so that we have [compare Eq. (2.5)]

$$w(t) = v_0 , \qquad (3.13a)$$

$$h(t,t') = 1 . \qquad (3.13b)$$

Now

$$x(t) = x_0 + \int_0^t dt'' v(t'')$$
$$= x_0 + v_0 t + \int_0^t dt' \int_0^{t'} dt'' R(t'') .$$

According to a well-known theorem [13],

$$\int_0^t dt' \int_0^{t'} dt'' R(t'') = \int_0^t dt'(t-t')R(t') , \qquad (3.14)$$

so that

$$u(t) = x_0 + v_0 t ,$$
$$g(t,t') = t - t' \quad (t' < t) . \qquad (3.15)$$

Let us now take $R(t)$ to be an OU noise, defined by [12,9]

$$P_0(y_1) = \left[\frac{\tau}{\pi f_0^2} \right]^{1/2} \exp \left[-\frac{\tau}{f_0^2}y^2 \right] , \qquad (3.16a)$$

$$T_{\Delta t}(y_{i+1}|y_i) = \left[\frac{\tau}{\pi f_0^2[1-\mu^2(\Delta t)]} \right]^{1/2} \qquad (3.16b)$$
$$\times \exp \left\{ -\frac{\tau[(y_{i+1}-\mu(\Delta t)y_i)]^2}{f_0^2[1-\mu^2(\Delta t)]} \right\} ,$$

$$\mu(\Delta t) = \exp \left[-\frac{\Delta t}{\tau} \right] , \qquad (3.16c)$$

where f_0 is a parameter describing the intensity of the random force.

The Fourier transforms $\hat{P}_0$ and $\hat{T}$ are [see Eqs. (3.4) and (3.9)]

$$\hat{P}_0(\zeta) = \exp \left[-\frac{f_0^2}{4\tau}\zeta^2 \right] , \qquad (3.17a)$$

$$\hat{T}_{\Delta t}(\xi) = \exp \left[\frac{(1-\mu^2)f_0^2}{4\tau}\xi^2 \right] . \qquad (3.17b)$$

From Eq. (3.11f), we find

$$a = -\frac{1}{\tau} , \qquad (3.18)$$

so that $\Psi(z)$ is now [from Eq. (3.11e)]

$$\Psi(z) = \exp \left[-\frac{z}{\tau} \right] . \qquad (3.19)$$

From the definition of $B(\xi)$, Eq. (3.11b), we have

$$B(\xi) = -\frac{f_0^2}{2\tau}\xi^2 . \qquad (3.20)$$

The details of the remaining algebra are in Appendix A. The final result is

$$\hat{p}(k,\theta,t) = \exp\{i[k(x_0+v_0t)+\theta v_0]$$
$$-\tfrac{1}{2}[\alpha(t)k^2+2\beta(t)k\theta+\gamma(t)\theta^2]\} , \qquad (3.21a)$$

where

$$\alpha(t) = f_0^2 \left[\frac{t^3}{3} - \frac{1}{2}t^2\tau + \tau^3(1-e^{-t/\tau}) - \tau^2 t e^{-t/\tau} \right] , \qquad (3.21b)$$

$$\beta(t) = \frac{1}{2}f_0^2[t^2 - \tau t(1-e^{-t/\tau})] , \qquad (3.21c)$$

$$\gamma(t) = f_0^2[t - \tau(1-e^{-t/\tau})] . \qquad (3.21d)$$

Since this is a Gaussian Fourier transform, it can be inverted easily to yield the real space joint PDF:

$$p(x,v,t)=\frac{1}{2\pi\sqrt{(\alpha\gamma-\beta^2)}}\exp\left\{-\frac{1}{2(\alpha\gamma-\beta^2)}[\gamma(x-x_0-v_0t)^2-2\beta(x-x_0-v_0t)(v-v_0)+\alpha(v-v_0)^2]\right\}, \qquad (3.22)$$

with α, β, and γ as defined in Eq. (3.21).

The marginal distributions for the position, $p(x,t)$, and for the velocity, $p(v,t)$, are most easily obtained from Eq. (3.21) by setting $k=0$ and $\theta=0$, respectively, then Fourier inverting. The results are

$$p(x,t)=\frac{1}{\sqrt{2\pi\alpha(t)}}\exp\left[-\frac{(x-x_0-v_0t)^2}{2\alpha(t)}\right], \qquad (3.23a)$$

$$p(v,t)=\frac{1}{\sqrt{2\pi\gamma(t)}}\exp\left[-\frac{(v-v_0)^2}{2\gamma(t)}\right]. \qquad (3.23b)$$

Equation (3.12) with an OU noise was investigated by Heinrichs [9], who derived Fokker-Planck (FP) equations for the joint PDF as well as for the two marginal PDF's $p(x,t)$ and $p(v,t)$. He solved these equations approximately only in two limiting cases: For very small τ (near $\tau=0$, which corresponds to white noise) and for very long τ (near $1/\tau=0$, which correspond to a constant force).

Here, however, we have obtained exact results in Eqs. (3.22) and (3.23), quite easily and without solving any differential equation. These expressions reduce to Heinrich's results in the appropriate limits. Also, a simple substitution shows that Eqs. (3.22) and (3.23) are indeed solutions of Heinrich's FP equations [9]. This further confirms the effectiveness of the present approach.

IV. GAUSSIAN PROCESSES

Another class of random noises that can be solved completely is the group of Gaussian noises. Except for the OU noise, all these processes are non-Markovian, and they are defined by the requirement that all the distribution functions $W_n(y_1,\ldots,y_n)$ be of Gaussian form. For zero-mean noises, this is equivalent (e.g., Ref. [7]) to the requirement that their Fourier transform be given by

$$\hat{W}(\varphi_1,\ldots,\varphi_n)=\exp\left[-\tfrac{1}{2}\sum_{i,j=1}^{n}\varphi_i\varphi_j\phi(i,j)\right], \qquad (4.1)$$

where $\phi(i,j)$ is the correlation function of the noise at time steps i and j, i.e., $\langle y_iy_j\rangle$. Note that formula (4.1) assumes that $\phi(i,j)=\phi(j,i)$.

Substituting (4.1) into (2.11), we have

$$\hat{p}(k,\theta,n)=\exp\left\{i(ku_n+\theta w_n)-\tfrac{1}{2}\left[k^2\sum_{i,j=0}^{n}g(n,i)g(n,j)\phi(i,j)\Delta t^2+k\theta\sum_{i,j=0}^{n}[g(n,i)h(n,j)+h(n,i)g(n,j)]\phi(i,j)\Delta t^2\right.\right.$$

$$\left.\left.+\theta^2\sum_{i,j=0}^{n}h(n,i)h(n,j)\phi(i,j)\Delta t^2\right]\right\}. \qquad (4.2)$$

Upon taking the limit $n\to\infty,\Delta t\to 0$, all the sums become integrals. Hence we have from Eq. (2.13)

$$\hat{p}(k,\theta,t)=\exp\{i[ku(t)+\theta w(t)]$$
$$-\tfrac{1}{2}[\alpha(t)k^2+2\beta(t)k\theta+\gamma(t)\theta^2]\}, \qquad (4.3a)$$

where

$$\alpha(t)=\int_0^t dt'\int_0^t dt''g(t,t')g(t,t'')\phi(t',t''), \qquad (4.3b)$$

$$\beta(t)=\tfrac{1}{2}\int_0^t dt'\int_0^t dt''[g(t,t')h(t,t'')$$
$$+h(t,t')g(t,t'')]\phi(t',t''), \qquad (4.3c)$$

$$\gamma(t)=\int_0^t dt'\int_0^t dt''h(t,t')h(t,t'')\phi(t',t''), \qquad (4.3d)$$

and $\phi(t',t'')=\langle R(t')R(t'')\rangle$ is the continuous time correlation function of the noise $R(t)$.

As an application of Eq. (4.3) and as a consistency check, we can use Eq. (3.12) again, $\ddot{x}=R(t)$, with $R(t)$ as OU noise (since this noise is both Markovian and Gaussian). For this case

$$\phi(t',t'')=\frac{f_0^2}{2\tau}\exp\left[-\frac{|t'-t''|}{\tau}\right], \qquad (4.4a)$$

$$g(t,t')=t-t' \quad (t>t'), \qquad (4.4b)$$

$$h(t,t')=1. \qquad (4.4c)$$

The functions α, β, and γ calculated through Eq. (4.3) are identical (as expected) with those obtained for the same noise (OU) in Eq. (3.21).

V. DIFFERENTIAL EQUATIONS FOR PDF'S IN THE GAUSSIAN CASE

The preceding sections presented cases where direct calculation of the PDF could be performed, i.e., when the limit in Eq. (2.13) could be calculated explicitly. In some cases, however, the limit proves too complicated for explicit calculation. Equation (2.11) then offers an alternative point of view by providing a general approach for deriving FP-like differential equations for the PDF's.

To exemplify this approach through a simple example, consider the Gaussian random noises described in Sec. IV. Having arrived at Eq. (4.2), we now choose not to take the limit $n\to\infty,\Delta t\to 0$ which would yield Eqs. (4.3). Instead, let us calculate the time derivative of $\hat{p}(k,\theta,t)$ from its discrete form $\hat{p}(k,\theta,n)$. This is just

$$\frac{\partial \hat{p}(k,\theta,t)}{\partial t} = \lim_{\Delta t \to 0} \frac{\hat{p}(k,\theta,n+1) - \hat{p}(k,\theta,n)}{\Delta t} \equiv \lim_{\Delta t \to 0} \frac{\Delta \hat{p}}{\Delta t} \ .$$

(5.1)

With the help of Eq. (4.2) for $\hat{p}(k,\theta,n)$, this calculation is straightforward but tedious, and the details can be found in Appendix B. The final result is

$$\frac{\partial \hat{p}(k,\theta,t)}{\partial t} = \left\{ i\frac{du}{dt}k + i\frac{dw}{dt}\theta \right.$$
$$\left. - \frac{1}{2}\left[\frac{d\alpha}{dt}k^2 + 2\frac{d\beta}{dt}k\theta + \frac{d\gamma}{dt}\theta^2 \right] \right\}$$
$$\times \hat{p}(k,\theta,t) \ ,$$

(5.2)

where α, β, and γ are the functions defined in Eqs. (4.3).

Equation (5.2) can be Fourier inverted to yield

$$\frac{\partial p(x,v,t)}{\partial t} = -\frac{du}{dt}\frac{\partial p}{\partial x} - \frac{dw}{dt}\frac{\partial p}{\partial v}$$
$$+ \frac{1}{2}\left[\frac{d\alpha}{dt}\frac{\partial^2 p}{\partial x^2} + 2\frac{d\beta}{dt}\frac{\partial^2 p}{\partial x \partial v} + \frac{d\gamma}{dt}\frac{\partial^2 p}{\partial v^2} \right] \ .$$

(5.3)

Here again, one should note the high generality of these equations. The result obtained here is valid for a wide range of Langevin equations as well as for a wide range of random noises.

We can now apply this formalism to the simple equation $\ddot{x} = R(t)$ with the random force being the OU noise already considered in Sec. III for which we have found the PDF [Eq. (3.22)]. Then from (4.3) and (4.4) we have

$$\frac{d\gamma}{dt} = f_0^2[1 - e^{-t/\tau}] \ ,$$

(5.4a)

$$\frac{d\beta}{dt} = \tfrac{1}{2}f_0^2[t(2 - e^{-t/\tau}) - \tau(1 - e^{-t/\tau})] \ ,$$

(5.4b)

$$\frac{d\alpha}{dt} = f_0^2[t^2 - t\tau(1 - e^{-t/\tau})] \ .$$

(5.4c)

The remarkable thing about Eqs. (5.3) and (5.4) is that they *differ* from the FP equations derived by Heinrichs for the *same* stochastic process [9]. Instead of Eqs. (5.3) and (5.4), Heinrichs obtains (for the case he investigated, i.e., $x_0 = v_0 = 0$, hence $u = w = 0$)

$$\frac{\partial p(x,v,t)}{\partial t} = \left[-v\frac{\partial}{\partial x} - b(t)\frac{\partial^2}{\partial x \partial v} + a(t)\frac{\partial^2}{\partial v^2} \right]$$
$$\times p(x,v,t) \ ,$$

(5.5a)

where

$$a(t) = \frac{f_0^2}{2}[1 - e^{-t/\tau}] \ ,$$

(5.5b)

$$b(t) = \frac{f_0^2}{2}[(t + \tau)e^{-t/\tau} - \tau] \ .$$

(5.5c)

This equation is not wrong. Indeed, as mentioned before, the function $p(x,v,t)$ calculated in Eq. (3.22) *is* the solu-

tion of Heinrichs's equation (5.5), *as well* as the solution of Eq. (5.3) (both with the appropriate initial conditions). This means that the FP equation corresponding to a given Langevin equation is *not unique* (and the existence of two different equations means trivially that there is an infinite number of them). Rather, the form of the equation depends on the specific method used to derive it. All the various FP equations, however, are equivalent to each other through the specific mathematical form of the solution.

It may seem that this last observation, as well as having the explicit solution, renders all discussions about the equations unimportant. This may be true for the case of the unconstrained particle, which is discussed here, but the discussion is very relevant to cases of constrained particles.

A particle can be "constrained," e.g., by the addition of boundaries to the system (for example, absorbing boundaries are often added for calculating first passage times). In the theory of the classical FP equation [14], the presence of such boundaries does not change the equation. Rather, it implies new mathematical boundary conditions on $p(x,v,t)$ (these usually make the task of solving the equations much more difficult).

It is tempting to extend such reasoning to Fokker-Planck-*like* equations, such as Eqs. (5.3) or (5.5). Indeed, Heinrichs used precisely this approach to tackle the case of a particle driven by OU noise when two absorbing boundaries are present [15]. However, the nonuniqueness of the FP equation casts serious doubt on the correctness of such an approach. It is highly unlikely, for example, that Eqs. (5.3) and (5.5) will still be equivalent to each other when new boundary conditions are added to them. Furthermore, it is quite possible that neither of them is correct in the presence of boundaries. Indeed, Eq. (5.3) follows from the general Eq. (2.11), which assumes implicitly the absence of constraints in so far as it allows summation over *all* paths that lead to the adequate final position and velocity. On the other hand, Heinrichs derived Eq. (5.5) by applying some averaging identities to the stochastic differential equation (3.12) [9]. Under constraints, such averaging procedures may change and one cannot assume *a priori* that the FP equation remains the same (or indeed exists at all).

In other words, the nonuniqueness of the FP equations implies that, whenever we add boundaries to the problem, we must provide independent mathematical justification for any FP equation we choose to use. In the absence of such justification, the results obtained by adding boundary conditions to any FP equations cannot be trusted.

VI. DIFFERENTIAL EQUATIONS FOR PDF'S IN THE CASE OF MARKOVIAN DICHOTOMOUS NOISE

For Gaussian noises, we have both an explicit expression for the PDF and a FP-like equation Eq. (5.3). However, and even more importantly, the method described in the last section can be used in cases where the limit $n \to \infty, \Delta t \to 0$ proves too complex to perform explicitly. As an example of the usefulness of this approach, I dis-

cuss now a specific type of random driving force, the Markovian dichotomous noise, which is used, for example, in the theory of second-order Butterworth filters [16].

Masoliver [10,11] has treated two stochastic differential equations with this particular noise, i.e., $\ddot{x}=R(t)$ and $\ddot{x}+\gamma\dot{x}=R(t)$, and has obtained FP-like equations for $p(x,v,t)$, $p(x,t)$, and $p(v,t)$ in both cases. Deriving the equations requires much work in each case, yet the general formula (2.13) provides a unifying point of view from which all these equations follow naturally. In this section, I derive a generic differential equation for all the various PDF's, when the driving force is the Markovian dichotomous noise, but the form of the equation is still general. Masoliver's equations follow immediately as several particular cases of this general equation. Thus we shall obtain a powerful generalization as well as a compact form for all the cases studied.

The Markovian dichotomous noise is defined by

$$P_0(y_1)=\tfrac{1}{2}[\delta(y_1-a)+\delta(y_1+a)] \ , \tag{6.1a}$$

$$T_{\Delta t}(y|y')=\tfrac{1}{2}[f(\Delta t)\delta(y-y')+g(\Delta t)\delta(y+y')] \ , \tag{6.1b}$$

where

$$f(\Delta t)=1+\exp(-2\lambda\Delta t) \ , \tag{6.1c}$$

$$g(\Delta t)=1-\exp(-2\lambda\Delta t) \ . \tag{6.1d}$$

Thus the force alternates between two values only, a and $-a$, and the probability that a switch from one value to the other occurs in the interval $(t,t+dt)$ is $\lambda e^{-\lambda t}dt$ [10]. The parameter λ is the average time between switches.

The calculation of $\hat{p}(k,\theta,n)$ follows the lines described in Secs. II–IV. The details are in Appendix C. The final result is

$$\hat{p}(k,\theta,n)=\exp[i(ku_n+\theta w_n)]\sum_{k=1}^{n}(\tfrac{1}{2})^n f^{n-k-1}g^{k-1}[F_k^n(\varphi_1,\ldots,\varphi_n)+F_k^n(-\varphi_1,\ldots,-\varphi_n)] \ , \tag{6.2a}$$

where

$$\varphi_i=[kg(n,i)+\theta h(n,i)]\Delta t \ . \tag{6.2b}$$

The coefficients $F_k^n(\varphi_1,\ldots,\varphi_n)$ are determined recursively by

$$F_k^n(\varphi_1,\ldots,\varphi_n)$$
$$=e^{ia\varphi_1}[F_k^{n-1}(\varphi_2,\ldots,\varphi_n)+F_{k-1}^{n-1}(-\varphi_2,\ldots,-\varphi_n)] \ , \tag{6.3a}$$

with the conditions

$$F_1^1(\varphi_1)=e^{ia\varphi_1} \ , \tag{6.3b}$$

$$F_0^m(\varphi_1,\ldots,\varphi_n)=0 \ , \tag{6.3c}$$

$$F_{m+1}^m(\varphi_1,\ldots,\varphi_n)=0 \ . \tag{6.3d}$$

In principle, Eq. (6.2) is the full solution of the problem. However, the limit $n\to\infty$, $\Delta t\to 0$ cannot be taken explicitly in all generality. Nonetheless, Eq. (6.2) provides an approximation of $\hat{p}(k,\theta,t)$, which can be calculated by selecting a sufficiently high value of n.

In the following, however, I use Eq. (6.2) for deriving the differential equation obeyed by $\hat{p}(k,\theta,t)$ along the lines of Sec. V. In the present case, however, this procedure yields a second-order equation in the time derivatives, rather than a first-order one as in the case of Gaussian noise.

For the purpose of this derivation, I shall assume that $g(n,i)$ and $h(n,i)$ depend on the difference $n-i$ only [or, in the continuum version $g(t,t')=g(t-t')$]. This is usually true. Also, I assume that $u(t)=w(t)=0$ (the particle starts at rest from the origin). This, however, is no loss of generality, because if we are interested in other cases all we need to do is multiply the function $\hat{p}(k,\theta,t)$ obtained for the case $u(t)=w(t)=0$ by $\exp\{i[ku(t)+\theta w(t)]\}$.

The details of the calculations can be found in Appen-

dix D. The main results are the following. There is no first-order equation in the time derivatives. This is because, to first order in Δt, we find that

$$\Delta\hat{p}(k,\theta,n)=ia\varphi_1\sum_{k=1}^{n}(\tfrac{1}{2})^n f^{n-k-1}g^{k-1}$$
$$\times[F_k^n(\varphi_2,\ldots,\varphi_{n+1})$$
$$-F_k^n(-\varphi_2,\ldots,-\varphi_{n+1})] \ , \tag{6.4}$$

with φ_i having the same meaning as in Eq. (6.2), i.e., $\varphi_i=[kg(n,i)+\theta h(n,i)]\Delta t$. Comparing this expression with Eq. (6.2) reveals that $\Delta\hat{p}$ cannot be expressed in terms of $\hat{p}(k,\theta,n)$ only; hence we cannot write down a first-order equation.

We therefore look at the second derivative, for which we need to calculate

$$\Delta^2\hat{p}\equiv\hat{p}(k,\theta,n+2)+\hat{p}(k,\theta,n)-2\hat{p}(k,\theta,n+1) \ .$$

This expression is then expanded to second order in powers of Δt. As shown in Appendix D, we can then express $\Delta^2\hat{p}/\Delta t^2$ in terms of $\Delta\hat{p}/\Delta t$ and $\hat{p}$, so that in the limit $n\to\infty$, $\Delta t\to 0$ we obtain a second-order differential equation for $\hat{p}(k,\theta,n)$. Quoting from Appendix D, we have finally that

$$\frac{\partial^2\hat{p}(k,\theta,t)}{\partial t^2}+\left[2\lambda-\frac{1}{\varphi}\frac{d\varphi}{dt}\right]\frac{\partial\hat{p}(k,\theta,t)}{\partial t}$$
$$+a^2\varphi^2\hat{p}(k,\theta,t)=0 \ , \tag{6.5a}$$

where

$$\varphi\equiv kg(t)+\theta h(t) \ . \tag{6.5b}$$

The power of Eq. (6.5) lies in the generality of the functions $g(t)$ and $h(t)$ that appear in it. This is because Eq. (6.5) covers all linear stochastic processes driven by a Markovian dichotomous noise. Thus, although the noise

is fully determined, the dynamics remains arbitrary [within the confines set by Eq. (2.2)], and therefore Eq. (6.5) covers at once a wide range of stochastic processes.

Consider, for example, the simple equation $\ddot{x} = R(t)$, for which $g(t) = t$, $h(t) = 1$. Then, from Eq. (6.5) we have immediately for $\hat{p}(k, \theta, t)$

$$\frac{\partial^2 \hat{p}(k, \theta, t)}{\partial t^2} + \left[2\lambda - \frac{k}{kt + \theta} \right] \frac{\partial \hat{p}(k, \theta, t)}{\partial t}$$
$$+ a^2 (kt + \theta)^2 \hat{p}(k, \theta, t) = 0 . \quad (6.6)$$

The equations for the FT of the marginal distribution for the position, i.e., $\hat{p}(k, t)$, and for the velocity, i.e., $\hat{p}(\theta, t)$, follow at once by setting $\theta = 0$ and $k = 0$, respectively:

$$\frac{\partial^2 \hat{p}(k, t)}{\partial t^2} + \left[2\lambda - \frac{1}{t} \right] \frac{\partial \hat{p}(k, t)}{\partial t} + a^2 k^2 t^2 \hat{p}(k, t) = 0 ,$$
$$(6.7a)$$

$$\frac{\partial^2 \hat{p}(\theta, t)}{\partial t^2} + 2\lambda \frac{\partial \hat{p}(\theta, t)}{\partial t} + a^2 \theta^2 \hat{p}(\theta, t) = 0 . \quad (6.7b)$$

Equation (6.7b) is the FT of the telegraph equation for $p(v, t)$. The two Eqs. (6.7) were derived separately by Masoliver [10], along with a somewhat more complicated version of Eq. (6.7) (the complication being due mainly to notations). As Masoliver pointed out, k and θ do not appear independently in the equation for $\hat{p}(k, \theta, t)$, but do so only through the combination $kt + \theta$. This fact was not immediately apparent in his version of the equation for $\hat{p}(k, \theta, t)$, but it was used later to derive Eq. (6.6).

There is no obvious reason in Masoliver's work for the appearance of such a combination. Here, on the other hand, it appears as a natural consequence of the general formula, Eq. (2.13), in which k and θ are always bound in a well-defined combination [except where they relate to $u(t)$ and $w(t)$]. This also makes very clear the meaning of the particular coefficients in this combination (t and 1), as they are merely the Green's functions of the original stochastic equation.

Masoliver later tackled the more general equation $\ddot{x} + \gamma \dot{x} = R(t)$ [11]. Much work is needed to derive the FP equations for $\hat{p}(k, \theta, t)$, $\hat{p}(k, t)$, and $\hat{p}(\theta, t)$. Moreover, in Masoliver's approach, the equations for the marginal distributions do not follow immediately from the equation for $\hat{p}(k, \theta, t)$, but instead require still more work. In the present approach, on the other hand, the derivations are almost trivial. For the equation $\ddot{x} + \gamma \dot{x} = R(t)$, we have

$$g(t - t') = \frac{1}{\gamma} \{ 1 - \exp[-\gamma(t - t')] \} , \quad (6.8a)$$

$$h(t - t') = \exp[-\gamma(t - t')] \quad (6.8b)$$

(easily derived, e.g., from Laplace transforming the equation). Thus we have at once

$$\frac{\partial^2 \hat{p}(k, \theta, t)}{\partial t^2} + \left[2\lambda - \frac{\gamma(k - \gamma\theta)e^{-\gamma t}}{k - (k - \gamma\theta)e^{-\gamma t}} \right] \frac{\partial \hat{p}(k, \theta, t)}{\partial t}$$
$$+ \frac{a^2}{\gamma^2} [k - (k - \gamma\theta)e^{-\gamma t}]^2 \hat{p}(k, \theta, t) = 0 , \quad (6.9a)$$

$$\frac{\partial^2 \hat{p}(k, t)}{\partial t^2} + \left[2\lambda - \frac{\gamma e^{-\gamma t}}{1 - e^{-\gamma t}} \right] \frac{\partial \hat{p}(k, t)}{\partial t}$$
$$+ \frac{a^2 k^2}{\gamma^2} (1 - e^{-\gamma t})^2 \hat{p}(k, t) = 0 , \quad (6.9b)$$

$$\frac{\partial^2 \hat{p}(\theta, t)}{\partial t^2} + (2\lambda + \gamma) \frac{\partial \hat{p}(\theta, t)}{\partial t} + a^2 \theta^2 e^{-2\gamma t} \hat{p}(\theta, t) = 0 . \quad (6.9c)$$

These agree with Masoliver's equations, but are derived much more easily. Again, the fact that k and θ appear only through the combination

$$k[1 - \exp(-\gamma t)] + \theta\gamma \exp(-\gamma t)$$

is not evident in Masoliver's approach, while it finds a very natural explanation in the present framework.

VII. SUMMARY

The main thrust of this work is methodological and conceptual. Rather than concentrating on a specific process, I have presented a general approach to a large class of stochastic differential equations.

The basis of the method is the discretization of the process described by the equation into n time steps. Each realization of the process consists then of n steps, and its probability can be calculated from the elementary properties of the random noise. In this discrete approximation, the probability of finding the particle in a given state at time t is the sum of the probabilities of the realizations which lead to this final state. Taking $n \to \infty$ then yields the required probability for the continuous process.

This method is successful on at least two fronts. First, if the limit $n \to \infty$ can be calculated explicitly, we obtain a closed expression for the required probability distribution function. This turns out to be the case for all Gaussian processes and monovariant Markov processes. There is no reason to believe these exhaust the possibilities of this method. Quite likely, other processes can be completely solved in this way.

Second, the expressions obtained for the discrete approximation are useful for deriving Fokker-Planck equations for the probability distribution functions. For the case of the Markovian dichotomous noise, I have shown that there is a generic second-order FP equation, which covers all the possible stochastic differential equations to which the general method is applicable. Thus we obtain a unified point of view on the various FP equations this particular noise can generate. In particular, for a joint probability distribution involving more than one stochastic variable (e.g., position and velocity), the Fourier frequencies corresponding to the stochastic variables appear only in a well-specified linear combination. The coefficients in this combination are the Green's function attached to each stochastic variable. Thus an important mathematical property of the FP equation is made explicit and its origin is clarified. This was not obvious in other approaches which have been applied to this problem.

Another general point underscored by the present approach is that there may be several FP equations corre-

sponding to a single stochastic differential equation. This nonuniqueness may have important implications when we consider particles constrained, e.g., by various boundaries. The standard recipe in Fokker-Planck theory, i.e., adding some boundary conditions, is incomplete, since the various FP equations may yield different results when supplemented with the *same* boundary conditions. Thus we must be careful when applying FP equations derived for free particles to constrained cases. Justification is always needed when taking such a step.

Finally, the discrete approximation should be of interest for numerical estimates of the PDF. I have not tried to address this issue here, and concentrated on analytical applications instead, but this should be investigated by interested parties.

In the last few years, there seems to be somewhat more interest in stochastic equations with complex environments and simple noises than in equations with simple environments and complex noises. I feel at least part of the reason for this is that the latter cases have proved quite difficult mathematically. Certainly, many physical systems do exhibit complex random noises. The method presented here may simplify the mathematical difficulties to a considerable extent. I hope this will motivate interested researchers to use it and develop it further.

ACKNOWLEDGMENTS

Many thanks to S. Redner, P. Krapivsky, and E. Ben Naim for very instructive discussions and criticisms, and to C. Doering for his suggestions and the references he contributed.

APPENDIX A

We wish to take the limit $n \to \infty, \Delta t \to 0$ of Eq. (3.10), i.e.,

$$\hat{p}(k,\theta,n) = e^{i(u_n k + w_n \theta)} \hat{P}_0(\zeta) \prod_{i=2}^{n} \hat{T}(\xi_i) , \qquad (A1a)$$

where

$$\zeta \equiv k \left[\sum_{k=1}^{n} g(n,k)[\mu(\Delta t)]^{k-1} \Delta t \right]$$
$$+ \theta \left[\sum_{k=1}^{n} h(n,k)[\mu(\Delta t)]^{k-1} \Delta t \right] , \qquad (A1b)$$

$$\xi_i \equiv k \left[\sum_{k=i}^{n} g(n,k)[\mu(\Delta t)]^{k-i} \Delta t \right]$$
$$+ \theta \left[\sum_{k=i}^{n} h(n,k)[\mu(\Delta t)]^{k-i} \Delta t \right] . \qquad (A1c)$$

To make this step clearer, I will perform it in several stages.

First, the expression $[\mu(\Delta t)]^{k-i}$ seems to become ill defined in this limit. Let us therefore introduce the following quantity:

$$a \equiv \frac{d[\ln\mu(t)]}{dt} \bigg|_{t=0} = \frac{1}{\mu(t)} \frac{d\mu(t)}{dt} \bigg|_{t=0} . \qquad (A2)$$

Hence, to first order in Δt,

$$\mu(\Delta t) \approx \mu(0)\exp(a\Delta t) , \qquad (A3)$$

so that

$$[\mu(\Delta t)]^{k-i} \approx \mu(0)\exp[a(k-i)\Delta t] \equiv \Psi(k-i) . \qquad (A4)$$

This expression has a well-defined value in the limit $n \to \infty, \Delta t \to 0$, when the function Ψ becomes

$$\Psi(z) = \mu(0)\exp(az) . \qquad (A5)$$

Next, note that all sums of the form $\sum_{k=i}^{n} \cdots \Delta t$ will become integrals in this limit. Introducing an integration variable $s \equiv k\Delta t$, an initial time $t' \equiv i\Delta t$, and the final time $t \equiv n\Delta t$, and using the definition of Ψ, Eq. (A5), we have

$$\sum_{k=i}^{n} g(n,k)[\mu(\Delta t)]^{k-i} \Delta t \to \int_{t'}^{t} g(t,s)\Psi(s-t')ds . \qquad (A6)$$

Similarly,

$$\sum_{k=i}^{n} h(n,k)[\mu(\Delta t)]^{k-i} \Delta t \to \int_{t'}^{t} h(t,s)\Psi(s-t')ds , \qquad (A7a)$$

$$\sum_{k=1}^{n} g(n,k)[\mu(\Delta t)]^{k-1} \Delta t \to \int_{0}^{t} g(t,s)\Psi(s)ds , \qquad (A7b)$$

$$\sum_{k=1}^{n} h(n,k)[\mu(\Delta t)]^{k-1} \Delta t \to \int_{0}^{t} h(t,s)\Psi(s)ds , \qquad (A7c)$$

where in the last two lines, the difference between $(k-1)\Delta t$ and $k\Delta t$ is neglected, since it yields a vanishing correction of order Δt to the expressions on the RHS.

Finally, consider the product over all the T functions in Eq. (A1a). Following the idea used in Eqs. (A2)–(A5), we define

$$B(\xi) = \frac{d}{d\Delta t} [\ln\hat{T}_{\Delta t}(\xi)]_{\Delta t=0} . \qquad (A8)$$

From the definition of $\hat{T}_{\Delta t}(\xi)$, we have $\hat{T}_{\Delta t}(0) = 1$. Hence, in the limit $n \to \infty, \Delta t \to 0$,

$$\hat{T}_{\Delta t}(\xi) \approx \exp[B(\xi)\Delta t] . \qquad (A9)$$

Substituting Eq. (A9) into the expression $\prod_{i=2}^{n} \hat{T}(\xi_i)$ which appears in Eq. (A1a) yields

$$\prod_{i=2}^{n} \hat{T}(\xi_i) \approx \exp \left[\sum_{i=2}^{n} B(\xi_i)\Delta t \right] . \qquad (A10)$$

In the limit $n \to \infty, \Delta t \to 0$, we have, from Eq. (A1c), that

$$\xi_i \to \xi(t,t') \equiv k \int_{t'}^{t} g(t,s)\Psi(s-t')ds$$
$$+ \theta \int_{t'}^{t} h(t,s)\Psi(s-t')ds , \qquad (A11)$$

and therefore

$$\sum_{i=2}^{n} B(\xi_i)\Delta t \to \int_{0}^{t} dt' B(\xi(t,t')) . \qquad (A12)$$

Putting together Eqs. (A1), (A6), (A7), and (A12), we finally have that

$$\hat{p}(k,\theta,t)=e^{i[ku(t)+\theta w(t)]}\hat{P}_0(\zeta(t))$$

$$\times\exp\left\{\int_0^t dt' B(\xi(t,t'))\right\}, \quad \text{(A13a)}$$

where

$$\zeta(t)\equiv k\int_0^t g(t,s)\Psi(s)ds+\theta\int_0^t h(t,s)\Psi(s)ds, \quad \text{(A13b)}$$

$$\xi(t,t')\equiv k\int_{t'}^t g(t,s)\Psi(s-t')ds$$

$$+\theta\int_{t'}^t h(t,s)\Psi(s-t')ds, \quad \text{(A13c)}$$

As an example, let us now apply this general result to the specific case of $\ddot{x}=R(t)$. As noted in Eq. (3.15),

$$u(t)=x_0+v_0 t, \quad w(t)=v_0,$$
$$g(t,t')=t-t', \quad h(t,t')=1 \quad (t'<t). \quad \text{(A14)}$$

Let us now take $R(t)$ to be an OU noise, defined by [12,9]

$$P_0(y_1)=\left[\frac{\tau}{\pi f_0^2}\right]^{1/2}\exp\left[-\frac{\tau}{f_0^2}y^2\right], \quad \text{(A15a)}$$

$$T_{\Delta t}(y_{i+1}|y_i)=\left[\frac{\tau}{\pi f_0^2[1-\mu^2(\Delta t)]}\right]^{1/2}$$

$$\times\exp\left\{-\frac{\tau[(y_{i+1}-\mu(\Delta t)y_i)]^2}{f_0^2[1-\mu^2(\Delta t)]}\right\}, \quad \text{(A15b)}$$

$$\mu(\Delta t)=\exp\left[-\frac{\Delta t}{\tau}\right], \quad \text{(A15c)}$$

where f_0 is a parameter describing the intensity of the random force.

The Fourier transforms $\hat{P}_0$ and $\hat{T}$ are [see Eqs. (3.4) and (3.9)]

$$\hat{P}_0(\zeta)=\exp\left[-\frac{f_0^2}{4\tau}\zeta^2\right], \quad \text{(A16a)}$$

$$\hat{T}_{\Delta t}(\xi)=\exp\left[\frac{(1-\mu^2)f_0^2}{4\tau}\xi^2\right]. \quad \text{(A16b)}$$

From Eq. (A2), we find

$$a=-\frac{1}{\tau}, \quad \text{(A17)}$$

so that $\Psi(z)$ is now [from Eq. (A5)]

$$\Psi(z)=\exp\left[-\frac{z}{\tau}\right]. \quad \text{(A18)}$$

From the definition of $B(\xi)$, Eq. (A8), we have

$$B(\xi)=-\frac{f_0^2}{2\pi}\xi^2. \quad \text{(A19)}$$

After some algebra, we obtain

$$\int_{t'}^t h(t,s)\Psi(s-t')ds=\int_{t'}^t \exp\left[-\frac{(s-t')}{\tau}\right]ds$$

$$=\tau\left\{1-\exp\left[-\frac{(t-t')}{\tau}\right]\right\}, \quad \text{(A20a)}$$

$$\int_{t'}^t g(t,s)\Psi(s-t')ds$$

$$=\int_{t'}^t (t-s)\exp\left[-\frac{(s-t')}{\tau}\right]ds$$

$$=\tau\left\{t-t'+\tau\left[\exp\left[-\frac{(t-t')}{\tau}\right]-1\right]\right\}, \quad \text{(A20b)}$$

from which we have immediately (by setting $t'=0$)

$$\int_0^t h(t,s)\Psi(s)ds=\tau\left[1-\exp\left[-\frac{t}{\tau}\right]\right], \quad \text{(A21a)}$$

$$\int_0^t g(t,s)\Psi(s)ds=\tau\left\{t-\tau\left[1-\exp\left[-\frac{t}{\tau}\right]\right]\right\}. \quad \text{(A21b)}$$

Since

$$\zeta=k\int_0^t g(t,s)\Psi(s)ds+\theta\int_0^t h(t,s)\Psi(s)ds,$$

by substituting (A21) into (A16a) we obtain

$$\hat{P}_0(\zeta)=\exp\{-\tfrac{1}{2}[a(t)k^2+b(t)\theta^2+2c(t)k\theta]\}, \quad \text{(A22a)}$$

where

$$a(t)=\frac{f_0^2\tau}{2}\left\{t-\tau\left[1-\exp\left[-\frac{t}{\tau}\right]\right]\right\}^2, \quad \text{(A22b)}$$

$$b(t)=\frac{f_0^2\tau}{2}\left[1-\exp\left[-\frac{t}{\tau}\right]\right]^2, \quad \text{(A22c)}$$

$$c(t)=\frac{f_0^2\tau}{2}\left\{t\left[1-\exp\left[-\frac{t}{\tau}\right]\right]\right.$$

$$\left.-\tau\left[1-\exp\left[-\frac{t}{\tau}\right]\right]^2\right\}. \quad \text{(A22d)}$$

Similarly, we must substitute Eqs. (A19) and (A20) into the expression $\int_0^t dt' B(\xi(t,t'))$, where

$$\xi(t,t')=k\int_{t'}^t g(t,s)\Psi(s-t')ds$$

$$+\theta\int_{t'}^t h(t,s)\Psi(s-t')ds.$$

The integral over t' can be performed explicitly, and after some more algebra we have that

$$\int_0^t dt' B(\xi(t,t'))=-\tfrac{1}{2}[l(t)k^2+m(t)\theta^2+2n(t)k\theta], \quad \text{(A23a)}$$

with

$$l(t)=f_0^2\left[\frac{t^3}{3}+t\tau^2-t^2\tau+\frac{\tau^3}{2}\right.$$
$$\left.-2t\tau^2e^{-t/\tau}-\frac{\tau^3}{2}e^{-2t/\tau}\right],\qquad(\text{A23b})$$

$$m(t)=f_0^2\left[t-\frac{3}{2}\tau+2\tau e^{-t/\tau}-\frac{\tau}{2}e^{-2t/\tau}\right],\qquad(\text{A23c})$$

$$n(t)=f_0^2\left[\frac{t^2}{2}-t\tau+\frac{\tau^2}{2}-\tau^2e^{-t/\tau}\right.$$
$$\left.+\tau^2e^{-2t/\tau}+t\tau e^{-t/\tau}\right].\qquad(\text{A23d})$$

Substituting (A14), (A22), and (A23) into (A13) yields the final answer:

$$\hat{p}(k,\theta,t)=\exp\{i[k(x_0+v_0t)+\theta v_0]$$
$$-\tfrac{1}{2}[\alpha(t)k^2+2\beta(t)k\theta+\gamma(t)\theta^2]\},$$
$$(\text{A24a})$$

where

$$\alpha(t)=f_0^2\left[\frac{t^3}{3}-\tfrac{1}{2}t^2\tau+\tau^3(1-e^{-t/\tau})-\tau^2te^{-t/\tau}\right],$$
$$(\text{A24b})$$

$$\beta(t)=\frac{1}{2}f_0^2[t^2-\tau t(1-e^{-t/\tau})],\qquad(\text{A24c})$$

$$\gamma(t)=f_0^2[t-\tau(1-e^{-t/\tau})].\qquad(\text{A24d})$$

APPENDIX B

To calculate the discrete form of the time derivative of $\hat{p}(k,\theta,t)$, we first expand $\hat{p}(k,\theta,n+1)$ in powers of Δt, keeping only terms up to the first order. From the expression for $\hat{p}(k,\theta,n)$, Eq. (4.2), we have

$$\hat{p}(k,\theta,n+1)=\exp\left\{i(ku_{n+1}+\theta w_{n+1})-\frac{1}{2}\left[k^2\sum_{i,j=0}^{n+1}g(n+1,i)g(n+1,j)\phi(i,j)\Delta t^2\right.\right.$$
$$+k\theta\sum_{i,j=0}^{n+1}[g(n+1,i)h(n+1,j)+h(n+1,i)g(n+1,j)]\phi(i,j)\Delta t^2$$
$$\left.\left.+\theta^2\sum_{i,j=0}^{n+1}h(n+1,i)h(n+1,j)\phi(i,j)\Delta t^2\right]\right\}.\qquad(\text{B1})$$

Let us define $\partial_1 g$ as the partial derivative of g with respect to its first variable, i.e.,

$$\partial_1 g(t,t')\equiv\frac{\partial g(t,t')}{\partial t},\qquad(\text{B2})$$

so that $\partial_1 g(n,i)$ means

$$\partial_1 g(n,i)\equiv\frac{\partial g(t,t')}{\partial t}\bigg|_{\substack{t=n\Delta t\\t'=i\Delta t}}.\qquad(\text{B3})$$

A similar convention holds for the function $h(t,t')$. With this convention, and for the purpose of calculating the RHS of Eq. (5.1), we can approximate

$$g(n+1,i)g(n+1,j)\approx[g(n,i)+\partial_1 g(n,i)\Delta t]$$
$$\times[g(n,j)+\partial_1 g(n,j)\Delta t].\qquad(\text{B4})$$

Keeping terms only up to first order in Δt, we have that

$$g(n+1,i)g(n+1,j)$$
$$\approx g(n,i)g(n,j)+g(n,j)\partial_1 g(n,i)\Delta t$$
$$+g(n,i)\partial_1 g(n,j)\Delta t,\qquad(\text{B5a})$$

$$g(n+1,i)h(n+1,j)+h(n+1,i)g(n+1,j)$$
$$\approx g(n,i)h(n,j)+h(n,i)g(n,j)$$
$$+h(n,j)\partial_1 g(n,i)\Delta t+h(n,i)\partial_1 g(n,j)\Delta t$$
$$+g(n,j)\partial_1 h(n,i)\Delta t+g(n,i)\partial_1 h(n,j)\Delta t,\qquad(\text{B5b})$$

$$h(n+1,i)h(n+1,j)$$
$$\approx h(n,i)h(n,j)+h(n,j)\partial_1 h(n,i)\Delta t$$
$$+h(n,i)\partial_1 h(n,j)\Delta t,\qquad(\text{B5c})$$

$$u_{n+1}\approx u_n+\frac{du_n}{dt}\Delta t,\qquad(\text{B5d})$$

$$w_{n+1}\approx w_n+\frac{dw_n}{dt}\Delta t.\qquad(\text{B5e})$$

Hence, e.g.,

$$\sum_{i,j=0}^{n+1}g(n+1,i)h(n+1,j)\phi(i,j)\Delta t^2=\sum_{i,j=0}^{n}g(n,i)h(n,j)\phi(i,j)\Delta t^2+\sum_{i,j=0}^{n}[h(n,j)\partial_1 g(n,i)+g(n,i)\partial_1 h(n,j)]\phi(i,j)\Delta t^3$$
$$+\sum_{j=0}^{n}g(n+1,n)h(n,j)\phi(n,j)\Delta t^2+\sum_{i=0}^{n}g(n,i)h(n+1,n)\phi(i,n)\Delta t^2.\qquad(\text{B6})$$

Note that the last term in the LHS sum,

$$g(n+1,n+1)h(n+1,n+1)\phi(n+1,n+1)\Delta t^2,$$

is of second order in Δt and was therefore neglected.

Using Eqs. (B5), we have that to first order in Δt

$$\Delta\hat{p}(k,\theta,n)=\hat{p}(k,\theta,n)\left\{-1+\exp\left[i\left[k\frac{du(n)}{dt}+\theta\frac{dw(n)}{dt}\right]-\tfrac{1}{2}(Ak^2+2Bk\theta+C\theta^2)\right]\right\},\quad\text{(B7a)}$$

where

$$A=\sum_{i,j=0}^{n}2\partial_1 g(n,i)g(n,j)\phi(i,j)\Delta t^3+2\partial_1 g(n,n)\sum_{j=0}^{n}g(n,j)\phi(n,j)\Delta t^2,\quad\text{(B7b)}$$

$$\begin{aligned}B=&\tfrac{1}{2}\sum_{i,j=0}^{n}[\partial_1 g(n,i)h(n,j)+\partial_1 h(n,j)g(n,i)+\partial_1 g(n,j)h(n,i)+\partial_1 h(n,i)g(n,j)]\phi(i,j)\Delta t^3\\
&+\tfrac{1}{2}\partial_1 g(n,n)\left[\sum_{i=0}^{n}h(n,i)\phi(i,n)\Delta t^2+\sum_{j=0}^{n}h(n,j)\phi(n,j)\Delta t^2\right]\\
&+\tfrac{1}{2}\partial_1 h(n,n)\left[\sum_{j=0}^{n}g(n,i)\phi(i,n)\Delta t^2+\sum_{i=0}^{n}g(n,j)\phi(n,j)\Delta t^2\right],\end{aligned}\quad\text{(B7c)}$$

$$C=\tfrac{1}{2}\sum_{i,j=0}^{n}2\partial_1 h(n,i)h(n,j)\phi(i,j)\Delta t^3+2\partial_1 h(n,n)\sum_{j=0}^{n}h(n,j)\phi(n,j)\Delta t^2.\quad\text{(B7d)}$$

In the limit $\Delta t\to 0$, we can expand the exponents and keep only the leading terms. Thus

$$\Delta\hat{p}(k,\theta,n)=\hat{p}(k,\theta,n)\left[i\left[k\frac{du(n)}{dt}+\theta\frac{dw(n)}{dt}\right]-\tfrac{1}{2}(Ak^2+2Bk\theta+C\theta^2)\right],\quad\text{(B8)}$$

where A, B, and C are defined in Eqs. (B7b)–(B7d).

In the limit $n\to\infty$, $\Delta t\to 0$, all sums become integrals. Taking for example the coefficient of k^2, we have

$$\left[\sum_{i,j=0}^{n}2\partial_1 g(n,i)g(n,j)\phi(i,j)\Delta t^3+2\partial_1 g(n,n)\sum_{j=0}^{n}g(n,j)\phi(n,j)\Delta t^2\right]$$

$$\to\left\{\int_0^t dt'\int_0^t dt''[2\partial_1 g(t,t')g(t,t'')+2\partial_1 g(t,t'')g(t,t')]+\int_0^t dt'\partial_1 g(t,t)g(t,t')\phi(t,t')\right\}\Delta t.\quad\text{(B9a)}$$

The expression on the RHS turns out to be

$$\left\{\frac{d}{dt}\left[\int_0^t dt'\int_0^t dt''g(t,t')g(t,t'')\phi(t',t'')\right]\right\}\Delta t,\quad\text{(B9b)}$$

which is just $d\alpha(t)/dt$, where $\alpha(t)$ is

$$\alpha(t)=\int_0^t dt'\int_0^t dt''g(t,t')g(t,t'')\phi(t',t'').\quad\text{(B9c)}$$

This is identical to the definition of $\alpha(t)$ in Eq. (4.3b).

Similar reasoning applied to the other terms in Eq. (B8) leads us to finally rewrite it, in the limit $n\to\infty$, $\Delta t\to 0$, as

$$\begin{aligned}\frac{\partial\hat{p}(k,\theta,t)}{\partial t}=&\left\{i\frac{du}{dt}k+i\frac{dw}{dt}\theta\right.\\
&\left.-\frac{1}{2}\left[\frac{d\alpha}{dt}k^2+2\frac{d\beta}{dt}k\theta+\frac{d\gamma}{dt}\theta^2\right]\right\}\\
&\times\hat{p}(k,\theta,t),\end{aligned}\quad\text{(B10)}$$

where α, β, and γ are

$$\alpha(t)=\int_0^t dt'\int_0^t dt''g(t,t')g(t,t'')\phi(t',t''),\quad\text{(B11a)}$$

$$\begin{aligned}\beta(t)=&\tfrac{1}{2}\int_0^t dt'\int_0^t dt''[g(t,t')h(t,t'')\\
&+h(t,t')g(t,t'')]\phi(t',t''),\end{aligned}\quad\text{(B11b)}$$

$$\gamma(t)=\int_0^t dt'\int_0^t dt''h(t,t')h(t,t'')\phi(t',t'').\quad\text{(B11c)}$$

APPENDIX C

The function $\hat{p}(k,\theta,n)$ is given by

$$\hat{p}(k,\theta,n)=e^{i(u_n k+w_n\theta)}$$

$$\times\hat{W}(\varphi_1,\ldots,\varphi_n)|_{\varphi_i=[g(n,i)k+h(n,i)\theta]\Delta t}.\quad\text{(C1)}$$

For the Markovian dichotomous noise, we have

$$P_0(y_1)=\tfrac{1}{2}[\delta(y_1-a)+\delta(y_1+a)],\quad\text{(C2a)}$$

$$T_{\Delta t}(y|y')=\tfrac{1}{2}[f(\Delta t)\delta(y-y')+g(\Delta t)\delta(y+y')],\quad\text{(C2b)}$$

where

$$f(\Delta t) = 1 + \exp(-2\lambda\Delta t) , \tag{C2c}$$

$$g(\Delta t) = 1 - \exp(-2\lambda\Delta t) , \tag{C2d}$$

and

$$W_n(y_1, \ldots, y_n) = P_0(y_1) T_{\Delta t}(y_2|y_1) \cdots T_{\Delta t}(y_n|y_{n-1}) . \tag{C2e}$$

We must calculate the Fourier transform of this expression and substitute the appropriate expressions for the Fourier parameters φ_i. We now define the following two functions:

$$W_n^+(y_1, \ldots, y_n)$$
$$= \tfrac{1}{2}\delta(y_1 - a) T_{\Delta t}(y_2|y_1) \cdots T_{\Delta t}(y_n|y_{n-1}) . \tag{C3a}$$

$$W_n^-(y_1, \ldots, y_n)$$
$$= \tfrac{1}{2}\delta(y_1 + a) T_{\Delta t}(y_2|y_1) \cdots T_{\Delta t}(y_n|y_{n-1}) . \tag{C3b}$$

According to the definitions of $P_0(y_1)$ and W_n in Eq. (C2), we see that

$$W_n = W_n^+ + W_n^- . \tag{C4}$$

Going over to the Fourier transform of the various functions, we have, using Eq. (C2b),

$$\hat{W}_n^+(\varphi_1, \ldots, \varphi_n) = \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n \exp[i(y_1\varphi_1 + \cdots + y_n\varphi_n)]$$
$$\times \tfrac{1}{2}\delta(y_1 - a)[\tfrac{1}{2}f\delta(y_2 - y_1) + \tfrac{1}{2}g\delta(y_2 + y_1)] T_{\Delta t}(y_3|y_2) \cdots T_{\Delta t}(y_n|y_{n-1}) \tag{C5}$$

and a similar equation for $\hat{W}_n^-$. After performing the integration on y_1, we have

$$\hat{W}_n^+(\varphi_1, \ldots, \varphi_n) = \tfrac{1}{2}e^{ia\varphi_1} \int_{-\infty}^{\infty} dy_2 \cdots \int_{-\infty}^{\infty} dy_n \exp[i(y_2\varphi_2 + \cdots + y_n\varphi_n)][\tfrac{1}{2}f\delta(y_2 - a) + \tfrac{1}{2}g\delta(y_2 + a)]$$
$$\times T_{\Delta t}(y_3|y_2) \cdots T_{\Delta t}(y_n|y_{n-1}) , \tag{C6a}$$

$$\hat{W}_n^-(\varphi_1, \ldots, \varphi_n) = \tfrac{1}{2}e^{-ia\varphi_1} \int_{-\infty}^{\infty} dy_2 \cdots \int_{-\infty}^{\infty} dy_n \exp[i(y_2\varphi_2 + \cdots + y_n\varphi_n)][\tfrac{1}{2}f\delta(y_2 + a) + \tfrac{1}{2}g\delta(y_2 - a)]$$
$$\times T_{\Delta t}(y_3|y_2) \cdots T_{\Delta t}(y_n|y_{n-1}) . \tag{C6b}$$

Using the definitions of W^+ and W^-, Eq. (C3), we have

$$\hat{W}_n^+(\varphi_1, \ldots, \varphi_n) = \tfrac{1}{2}e^{ia\varphi_1}\{f\hat{W}_{n-1}^+(\varphi_2, \ldots, \varphi_n)$$
$$+ g\hat{W}_{n-1}^-(\varphi_2, \ldots, \varphi_n)\} , \tag{C7a}$$

$$\hat{W}_n^-(\varphi_1, \ldots, \varphi_n) = \tfrac{1}{2}e^{-ia\varphi_1}\{f\hat{W}_{n-1}^-(\varphi_2, \ldots, \varphi_n)$$
$$+ g\hat{W}_{n-1}^+(\varphi_2, \ldots, \varphi_n)\} , \tag{C7b}$$

with

$$\hat{W}_1^+ = \tfrac{1}{2}e^{ia\varphi_1} , \tag{C7c}$$

$$\hat{W}_1^- = \tfrac{1}{2}e^{-ia\varphi_1} . \tag{C7d}$$

We now have the following lemma.
Lemma.

$$\hat{W}_n^+(\varphi_1, \ldots, \varphi_n) = \hat{W}_n^-(-\varphi_1, \ldots, -\varphi_n) .$$

The proof follows immediately from (C7) by induction on n.

The first terms in the series $\{\hat{W}_n^+\}_{n=1}^{\infty}$ are

$$\hat{W}_1^+ = \tfrac{1}{2}e^{ia\varphi_1} , \tag{C8a}$$

$$\hat{W}_2^+ = (\tfrac{1}{2})^2[fe^{ia(\varphi_1 + \varphi_2)} + ge^{ia(\varphi_1 - \varphi_2)}] , \tag{C8b}$$

$$\hat{W}_3^+ = (\tfrac{1}{2})^3[f^2 e^{ia(\varphi_1 + \varphi_2 + \varphi_3)} + fge^{ia(\varphi_1 + \varphi_2 - \varphi_3)}$$
$$+ fge^{ia(\varphi_1 - \varphi_2 - \varphi_3)} + g^2 e^{ia(\varphi_1 - \varphi_2 + \varphi_3)}] ,$$
$$\vdots \tag{C8c}$$

This suggests that we can write

$$\hat{W}_n^+(\varphi_1, \ldots, \varphi_n) = \sum_{k=1}^{n} (\tfrac{1}{2})^n f^{n-k-1} g^{k-1} F_k^n(\varphi_1, \ldots, \varphi_n) , \tag{C9}$$

with F_k^n yet to be determined. Note that $\hat{W}_n^-(\varphi_1, \ldots, \varphi_n)$ follows from $\hat{W}_n^+(\varphi_1, \ldots, \varphi_n)$ by virtue of the above lemma.

Substituting the form (C9) into the Eqs. (C7) we find

$$\sum_{k=1}^{n} (\tfrac{1}{2})^n f^{n-k-1} g^{k-1} F_k^n(\varphi_1, \ldots, \varphi_n) = \tfrac{1}{2}fe^{ia\varphi_1} \sum_{j=1}^{n-1} (\tfrac{1}{2})^{n-1} f^{n-j-2} g^{j-1} F_j^{n-1}(\varphi_2, \ldots, \varphi_n)$$
$$+ \tfrac{1}{2}ge^{ia\varphi_1} \sum_{j=1}^{n-1} (\tfrac{1}{2})^{n-1} f^{n-j-2} g^{j-1} F_j^{n-1}(-\varphi_2, \ldots, -\varphi_n) . \tag{C10}$$

The first term on the RHS can be rewritten as

$$e^{ia\varphi_1}\sum_{k=1}^{n-1}(\tfrac{1}{2})^n f^{n-k-1}g^{k-1}F_k^{n-1}(\varphi_2,\ldots,\varphi_n)\ , \tag{C11a}$$

where the index j has been renamed k. The second term on the RHS of Eq. (C10) can be rewritten as

$$e^{ia\varphi_1}\sum_{k=2}^{n}(\tfrac{1}{2})^n f^{n-k-1}g^{k-1}F_{k-1}^{n-1}(-\varphi_2,\ldots,-\varphi_n)\ , \tag{C11b}$$

where the new summation index k is defined as $k=j+1$.

Comparing the two sides of Eq. (C10) term by term with the help of Eqs. (C11), we see that we must have

$$F_k^n(\varphi_1,\ldots,\varphi_n)=\begin{cases} e^{ia\varphi_1}F_1^{n-1}(\varphi_2,\ldots,\varphi_n),\quad k=1 \\ e^{ia\varphi_1}[F_k^{n-1}(\varphi_2,\ldots,\varphi_n)+F_{k-1}^{n-1}(-\varphi_2,\ldots,-\varphi_n)],\quad 1<k<n \\ e^{ia\varphi_1}F_{n-1}^{n-1}(-\varphi_2,\ldots,-\varphi_n),\quad k=n\ . \end{cases} \tag{C12}$$

The three cases in Eq. (C12) can be summed up as

$$F_k^n(\varphi_1,\ldots,\varphi_n)=e^{ia\varphi_1}[F_k^{n-1}(\varphi_2,\ldots,\varphi_n)+F_{k-1}^{n-1}(-\varphi_2,..,-\varphi_n)]\ , \tag{C13a}$$

if we add the conventions

$$F_0^m(\varphi_1,\ldots,\varphi_n)=0\ , \tag{C13b}$$

$$F_{m+1}^m(\varphi_1,\ldots,\varphi_n)=0\ . \tag{C13c}$$

Finally, we note that from $\hat{W}_1^+=\tfrac{1}{2}e^{ia\varphi_1}$ we have that

$$F_1^1(\varphi_1)=e^{ia\varphi_1}\ . \tag{C14}$$

Equations (C13) and (C14) fully determine the coefficients $F_k^n(\varphi_1,\ldots,\varphi_n)$. Equations (C1), (C4), (C9), and the lemma now allow us to write the function $\hat{p}(k,\theta,n)$ as

$$\hat{p}(k,\theta,n)=\exp[i(ku_n+\theta w_n)]\sum_{k=1}^{n}(\tfrac{1}{2})^n f^{n-k-1}g^{k-1}[F_k^n(\varphi_1,\ldots,\varphi_n)+F_k^n(-\varphi_1,\ldots,-\varphi_n)]|_{\varphi_i=[kg(n,i)+\theta h(n,i)]\Delta t}\ . \tag{C15}$$

APPENDIX D

We wish to calculate the discrete version of the first derivative, $\Delta\hat{p}(k,\theta,n)$, defined as $\hat{p}(k,\theta,n+1)-\hat{p}(k,\theta,n)$, with $\hat{p}(k,\theta,n)$ defined by Eq. (C15). Rewriting (C15) with n replaced by $n+1$ yields

$$\hat{p}(k,\theta,n+1)=\sum_{k=1}^{n+1}\{(\tfrac{1}{2})^{n+1}f^{n-k}g^{k-1}[F_k^{n+1}(\varphi_1,\ldots,\varphi_{n+1})+F_k^{n+1}(-\varphi_1,\ldots,-\varphi_{n+1})]\}$$

$$=\sum_{k=1}^{n+1}(\tfrac{1}{2})^{n+1}f^{n-k}g^{k-1}\{e^{ia\varphi_1}[F_k^n(\varphi_2,\ldots,\varphi_{n+1})+F_k^n(-\varphi_2,\ldots,-\varphi_{n+1})]$$

$$+e^{-ia\varphi_1}[F_k^n(-\varphi_2,\ldots,-\varphi_{n+1})+F_k^n(\varphi_2,\ldots,\varphi_{n+1})]\}\ , \tag{D1}$$

where we have used the recurrence relations for F_k^n, Eqs. (C13). In Eq. (D1), $\varphi_i=[kg(n+1-i)+\theta h(n+1-i)]\Delta t$, in accordance with our assumption that $g(n,i)=g(n-i)$.

Remembering that $F_{n+1}^n(\varphi_2,\ldots,\varphi_{n+1})=0$ [see Eq. (C13c)], we can rewrite the first and third terms in the RHS of Eq. (D1) as

$$\sum_{k=1}^{n+1}(\tfrac{1}{2})^{n+1}f^{n-k}g^{k-1}[e^{ia\varphi_1}F_k^n(\varphi_2,\ldots,\varphi_{n+1})+e^{-ia\varphi_1}F_k^n(-\varphi_2,\ldots,-\varphi_{n+1})]$$

$$=\sum_{k=1}^{n}(\tfrac{1}{2})^n f^{n-k-1}g^{k-1}\frac{f}{2}[e^{ia\varphi_1}F_k^n(\varphi_2,\ldots,\varphi_{n+1})+e^{-ia\varphi_1}F_k^n(-\varphi_2,\ldots,-\varphi_{n+1})]\ . \tag{D2}$$

In the remaining two terms in the RHS of Eq. (D1), we change the summation index from k to $j=k-1$. Remembering that $F_0^n(\varphi_2,\ldots,\varphi_{n+1})=0$ [see Eq. (C13b)], we have

$$\sum_{k=1}^{n+1} (\tfrac{1}{2})^{n+1} f^{n-k} g^{k-1} [e^{ia\varphi_1} F_k^n(-\varphi_2, \ldots, -\varphi_{n+1}) + e^{-ia\varphi_1} F_k^n(\varphi_2, \ldots, \varphi_{n+1})]$$

$$= \sum_{j=1}^{n} (\tfrac{1}{2})^n f^{n-j-1} g^{j-1} \frac{g}{2} [e^{-ia\varphi_1} F_j^n(\varphi_2, \ldots, \varphi_{n+1}) + e^{ia\varphi_1} F_j^n(-\varphi_2, \ldots, -\varphi_{n+1})] . \quad \text{(D3)}$$

Combining (D2) and (D3), we obtain

$$\hat{p}(k, \theta, n+1) = \sum_{k=1}^{n} (\tfrac{1}{2})^n f^{n-k-1} g^{k-1} \left\{ \frac{f}{2} [e^{ia\varphi_1} F_k^n(\varphi_2, \ldots, \varphi_{n+1}) + e^{-ia\varphi_1} F_k^n(-\varphi_2, \ldots, -\varphi_{n+1})] \right.$$

$$\left. + \frac{g}{2} [e^{-ia\varphi_1} F_k^n(\varphi_2, \ldots, \varphi_{n+1}) + e^{ia\varphi_1} F_k^n(-\varphi_2, \ldots, -\varphi_{n+1})] \right\} , \quad \text{(D4)}$$

with $\varphi_i = kg(n-i+1) + \theta h(n+1-i)$. From this we subtract

$$\hat{p}(k, \theta, n) = \sum_{k=1}^{n} (\tfrac{1}{2})^n f^{n-k-1} g^{k-1} [F_k^n(\varphi_1', \ldots, \varphi_n') + F_k^n(-\varphi_1', \ldots, -\varphi_n')] , \quad \text{(D5)}$$

where $\varphi_i' = kg(n-i) + \theta h(n-i)$. Note now that $\varphi_{i+1} = \varphi_i'$ [it is for this relation that we assumed that $g(n,i) = g(n-i)$ and $h(n,i) = h(n-i)$]. Hence $\Delta\hat{p}$ can be written as

$$\Delta\hat{p} = \sum_{k=1}^{n} (\tfrac{1}{2})^n f^{n-k-1} g^{k-1} \left\{ F_k^n(\varphi_2, \ldots, \varphi_{n+1}) \left[\frac{f}{2} e^{ia\varphi_1} + \frac{g}{2} e^{-ia\varphi_1} - 1 \right] \right.$$

$$\left. + F_k^n(-\varphi_2, \ldots, -\varphi_n) \left[\frac{f}{2} e^{-ia\varphi_1} + \frac{g}{2} e^{ia\varphi_1} - 1 \right] \right\} . \quad \text{(D6)}$$

Since ultimately we want the limit $\Delta t \to 0$ of $\Delta\hat{p}/\Delta t$, we now expand the expressions in square brackets up to first order in Δt. Referring to Eq. (C2), we have

$$f(\Delta t) = 2 - 2\lambda\Delta t + 2\lambda^2 \Delta t^2 + \cdots , \quad \text{(D7a)}$$

$$g(\Delta t) = 2\lambda\Delta t - 2\lambda^2 \Delta t^2 + \cdots , \quad \text{(D7b)}$$

$$\exp(ia\varphi_1) = 1 + ia\varphi_1 - \tfrac{1}{2} a^2 \varphi_1^2 + \cdots , \quad \text{(D7c)}$$

where the last line follows from the fact that φ_1 is first order in Δt, because $\varphi_1 = [kg(n) + \theta h(n)]\Delta t$.

Hence, to first order in Δt,

$$\Delta\hat{p}(k, \theta, n) = ia\varphi_1 \sum_{k=1}^{n} (\tfrac{1}{2})^n f^{n-k-1} g^{k-1}$$

$$\times [F_k^n(\varphi_2, \ldots, \varphi_{n+1})$$

$$- F_k^n(-\varphi_2, \ldots, -\varphi_{n+1})] . \quad \text{(D8)}$$

This expression is not related in any obvious way to $\hat{p}(k, \theta, n)$ and therefore we cannot write a first-order differential equation for $\hat{p}(k, \theta, n)$. Let us therefore look at the second derivative of $\hat{p}(k, \theta, n)$, the discrete form of which is $\Delta^2\hat{p}/\Delta t^2$, where $\Delta^2\hat{p}$ is given by

$$\hat{p}(k, \theta, n+2) + \hat{p}(k, \theta, n) - 2\hat{p}(k, \theta, n+1) .$$

The expression $\hat{p}(k, \theta, n+2)$ is calculated in much the same way as $\hat{p}(k, \theta, n+1)$ in Eqs. (D1)–(D4). This time, we will need to use the recurrence relation Eq. (C13) twice. The algebra is straightforward, and we obtain finally

$$\hat{p}(k, \theta, n+2) = \sum_{k=1}^{n} (\tfrac{1}{2})^n f^{n-k-1} g^{k-1}$$

$$\times \left\{ F_k^n(\varphi_2, \ldots, \varphi_{n+1}) \left[\frac{f^2}{4} e^{ia(\varphi_0 + \varphi_1)} + \frac{fg}{4} e^{ia(\varphi_1 - \varphi_0)} + \frac{fg}{4} e^{-ia(\varphi_1 + \varphi_0)} + \frac{g^2}{4} e^{ia(\varphi_0 - \varphi_1)} \right] \right.$$

$$\left. + F_k^n(-\varphi_2, \ldots, -\varphi_n) \left[\frac{f^2}{4} e^{-ia(\varphi_0 + \varphi_1)} + \frac{fg}{4} e^{ia(\varphi_0 - \varphi_1)} + \frac{fg}{4} e^{ia(\varphi_0 + \varphi_1)} + \frac{g^2}{4} e^{ia(\varphi_1 - \varphi_0)} \right] \right\} , \quad \text{(D9)}$$

where, for the sake of consistency, $\varphi_1, \ldots, \varphi_{n+1}$ are the *same* as in Eq. (D1), i.e., $\varphi_i = [kg(n-i+1) + \theta h(n-i+1)\Delta t$, and we have defined $\varphi_0 = [kg(n+1) + \theta h(n+1)]\Delta t$.

Substituting Eqs. (D4), (D5), and (D9) into the expression for $\Delta^2\hat{p}$ and expanding to second order in Δt [using Eq. (D7)], we finally obtain

$$\Delta^2\hat{p}(k,\theta,n)=ia\left[(\varphi_0-\varphi_1)-2\lambda\varphi_0\Delta t\right]\left\{\sum_{k=1}^{n}(\tfrac{1}{2})^n f^{n-k-1}g^{k-1}[F_k^n(\varphi_2,\ldots,\varphi_{n+1})-F_k^n(-\varphi_2,\ldots,-\varphi_{n+1})]\right\}$$

$$-a^2\varphi_1^2\sum_{k=1}^{n}(\tfrac{1}{2})^n f^{n-k-1}g^{k-1}[F_k^n(\varphi_2,\ldots,\varphi_{n+1})+F_k^n(-\varphi_2,\ldots,-\varphi_{n+1})]\;. \tag{D10}$$

Note that $\varphi_0-\varphi_1$ is indeed of second order in Δt, as it should be, because

$$\varphi_0-\varphi_1=\{k[g(n+1)-g(n)]$$
$$+\theta[h(n+1)-h(n)]\}\Delta t\;,$$

and $g(n+1)-g(n)$ is already of first order in Δt. Comparing Eq. (D10) with Eqs. (D5) and (D8), we have that

$$\frac{\Delta^2\hat{p}(k,\theta,n)}{\Delta t^2}=-\frac{a^2\varphi_1^2}{\Delta t^2}\hat{p}(k,\theta,n)+\left[\frac{\varphi_0-\varphi_1-2\lambda\varphi_0\Delta t}{\varphi_1\Delta t}\right]$$
$$\times\frac{\Delta\hat{p}(k,\theta,n)}{\Delta t}\;. \tag{D11}$$

Denoting $\varphi\equiv kg(t)+\theta h(t)$, we see that

$$\frac{\varphi_1^2}{\Delta t^2}=[kg(n)+\theta h(n)]^2\to\varphi^2\;, \tag{D12a}$$

$$\frac{\varphi_0-\varphi_1}{\varphi_1\Delta t}=\frac{1}{\varphi_1}\left[k\frac{g(n+1)-g(n)}{\Delta t}+\theta\frac{h(n+1)-h(n)}{\Delta t}\right]$$
$$\to\frac{1}{\varphi}\frac{d\varphi}{dt}\;, \tag{D12b}$$

$$\frac{\varphi_0\Delta t}{\varphi_1\Delta t}\to 1\;, \tag{D12c}$$

where we have used the fact that $\varphi_0,\varphi_1\to\varphi$ when $n\to\infty,\Delta t\to 0$. Hence, in the limit $n\to\infty,\Delta t\to 0$, Eq. (D10) finally becomes

$$\frac{\partial^2\hat{p}(k,\theta,t)}{\partial t^2}+\left[2\lambda-\frac{1}{\varphi}\frac{d\varphi}{dt}\right]\frac{\partial\hat{p}(k,\theta,t)}{\partial t}$$
$$+a^2\varphi^2\hat{p}(k,\theta,t)=0\;, \tag{D13a}$$

where

$$\varphi\equiv kg(t)+\theta h(t)\;. \tag{D13b}$$

[1] The classic review of the standard theory is S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943), reprinted in *Selected Papers on Noise and Stochastic Processes,* edited by N. Wax (Dover, New York, 1964).

[2] M. I. Dykman, R. Mannella, P. V. E. McClintock, S. M. Soskin, and N. G. Stocks, Phys. Rev. A **43**, 1701 (1991).

[3] M. I. Dykman, Phys. Rev. A **43**, 2020 (1989); M. I. Dykman, D. G. Luchinsky, P. V. E. McClintock, N. D. Stein, and N. G. Stocks, *ibid.* **46**, R1713 (1992).

[4] N. G. Van Kampen, J. Stat. Phys. **54**, 1289 (1989).

[5] *Noise in Nonlinear Dynamical Systems,* edited by F. Moss and P. V. E. McClintock (Cambridge University, Cambridge, England, 1989), Vol. I, and references therein. See particularly the contributions of J. M. Sancho and M. San Miguel (p. 72) and P. Hänggi (p. 307).

[6] See, e.g., B. J. Alder and T. E. Wainwright, Phys. Rev. Lett. **18**, 988 (1967); Phys. Rev. A **1**, 18 (1970).

[7] R. Kubo, M. Toda, and N. Hashitsume, *Statistical Physics II* (Springer-Verlag, Berlin, 1985).

[8] M. Araujo, S. Havlin, H. Larralde, and H. E. Stanley, Phys. Rev. Lett. **68**, 1791 (1992).

[9] J. Heinrichs, Phys. Rev. E **47**, 3007 (1993).

[10] J. Masoliver, Phys. Rev. A **45**, 706 (1992).

[11] J. Masoliver, Phys. Rev. E **48**, 7121 (1993).

[12] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry,* 2nd ed. (North-Holland, Amsterdam, 1992).

[13] e.g., R. Ghez, *A Primer of Diffusion Problems* (Wiley, New York, 1988), p. 187.

[14] H. Risken, *The Fokker-Planck Equation,* 2nd ed. (Springer-Verlag, Berlin, 1989).

[15] J. Heinrichs, Phys. Rev. E **48**, 2397 (1993).

[16] R. F. Pawula, Phys. Rev. A **35**, 3102 (1987); **36**, 4996 (1987).